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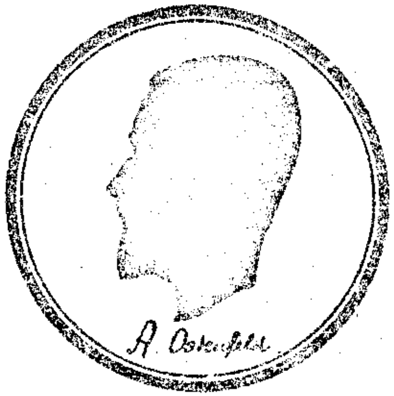
ELASTIC SEMI-INFINITE MEDIUM BOUNDED BY A RIGID WALL WITH A CIRCULAR HOLE

By

S. GRAVESEN



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INTRODUCTION

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In the present study a system will be discussed consisting of an homogeneous and isotropic elastic semi-infinite medium, bounded by an infinitely rigid wall with a circular hole. For convenience, the wall will be considered as lying horizontally. The system will be subjected to vertical loads, acting internal to the hole and having symmetry around its centre (see fig. 1).

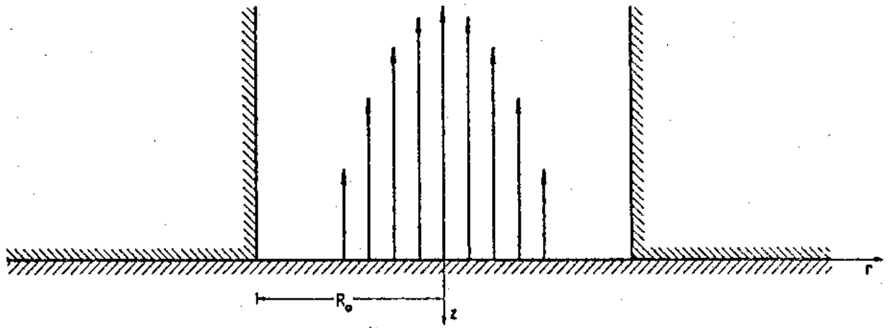

Fig. 1. Elastic medium bounded by a rigid wall with a circular hole and subjected to an axially symmetric load.

It is assumed that overall contact is maintained between wall and semi-infinite medium, which at any rate is the case if the necessary stresses between the elastic medium and the wall can be transmitted, or, as we shall see later, provided the applied loading in the hole consists of normal tensions throughout. With respect to the friction along the surface of the semi-infinite medium, it will be assumed either that shear stresses cannot be transmitted between the elastic medium and

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the wall, or that the surface of the elastic medium is constrained with respect to horizontal movements both at the wall and at the hole.

Our task, then, is to find the vertical displacements of the surface of the elastic medium within the circular hole, and the normal stresses between the semi-infinite medium and the wall. Any shear stresses arising between the semi-infinite medium and the wall, or horizontal movements of the surface, will, however, not be determined.

The problem is solved by first considering the load in question as acting on the semi-infinite elastic medium without the bounding wall, and calculating the deflections hereby produced. The normal stresses between the elastic medium and the wall can then be determined, in that the deflections produced by these stresses, and the deflections resulting from the external loading, have to cancel each other out in the region exterior to the hole. The vertical deflections sought can then be found as the sum of the deflections arising from the applied load and from the normal stresses between the wall and the elastic medium.

It is however unneccessary to find the stresses between the elastic medium and the wall, in order to determine the deflections which they produce within the hole. These deflections can in fact be expressed direct in terms of the deflections which the same stresses produce in the region exterior to the hole, which latter deflections are in fact equal to those produced by the external loading, with opposite sign.

In what immediately follows, we will consider an elastic medium without bounding wall, and either with a completely free surface, described as case *a*, or with the surface constrained with respect to horizontal movements, described as case *b*.

We introduce cylindrical coordinates (r, θ, z), so that the plane $z = 0$ constitutes the boundary of the elastic medium, this lying on the side of the positive z-values. The distance from the point of origin to an arbitrary point (r, θ, z) will be designated by

$$s = \sqrt{r^2 + z^2}.$$

The displacements in the z- and r-directions will be designated w and v respectively. As we are concerned exclusively with loads which have symmetry about the z-axis, there will not be any displacements in the θ-direction.

The semi-infinite medium is characterized by Young's modulus E and Poisson's ratio ν.

In the case where the semi-infinite medium has a free surface (case *a*), the displacements resulting from a single force P applied at the point of origin and directed along the z-axis will be (see e. g. Timoshenko [4])

$$w = \frac{P}{2\pi E}\left((1+\nu)\frac{z^2}{s^3} + 2(1-\nu^2)\frac{1}{s}\right),$$

$$v = \frac{P}{2\pi E}\frac{(1-2\nu)(1+\nu)}{r}\left(\frac{z}{s} - 1 + \frac{1}{1-2\nu}\frac{r^2 z}{s^3}\right),$$

and in particular for $z = 0$,

$$w = \frac{P(1-\nu^2)}{\pi E}\frac{1}{r},$$

$$v = -\frac{P(1-2\nu)(1+\nu)}{2\pi E}\frac{1}{r}.$$

If the surface of the semi-infinite medium is constrained with respec to movements in directions parallel to the plane $z = 0$ (case *b*), ther the displacements resulting from a single force P, applied as abov at the point of origin and in the direction of the z-axis, will accordin to Boussinesq [2] be

$$w = \frac{P}{4\pi E}\frac{1+\nu}{1-\nu}\left(\frac{z^2}{s^3} + (3-4\nu)\frac{1}{s}\right),$$

$$v = \frac{P}{4\pi E}\frac{1+\nu}{1-\nu}\frac{zr}{s^3},$$

and in particular for $z = 0$,

$$w = \frac{P}{4\pi E}\frac{(1+\nu)(3-4\nu)}{1-\nu}\frac{1}{r},$$

$$v = 0.$$

To constrain the surface with respect to horizontal movements is necessary to have the following radial shear stresses in the surface

$$\tau = \frac{P}{4\pi}\frac{1-2\nu}{1-\nu}\frac{1}{r^2}.$$

The displacements resulting from a uniformly distributed line load p, acting around a circle with centre at the point of origin and of radius R, are found by integrating the displacements from a single force. In case a we obtain

$$w_{z=0} = \frac{4(1-\nu^2)}{\pi E} pR \int_0^{\frac{\pi}{2}} \frac{d\omega}{\sqrt{R^2 - r^2 \sin^2\omega}} = \frac{4(1-\nu^2)}{\pi E} pK\left(\frac{r}{R}\right), \qquad r < R,$$

$$w_{z=0} = \frac{4(1-\nu^2)}{\pi E} pR \int_0^{\frac{\pi}{2}} \frac{d\omega}{\sqrt{r^2 - R^2 \sin^2\omega}} = \frac{4(1-\nu^2)}{\pi E} p\frac{R}{r} K\left(\frac{R}{r}\right), \qquad r > R,$$

$$v_{z=0} = 0, \qquad r < R,$$

$$v_{z=0} = -\frac{(1-2\nu)(1+\nu)}{E} p\frac{R}{r}, \qquad r > R.$$

The integrals designated by $K(k)$ are complete elliptic integrals of the 1st kind. In case b, and with the same load as above, we have

$$w_{z=0} = \frac{(1+\nu)(3-4\nu)}{\pi E(1-\nu)} pK\left(\frac{r}{R}\right), \qquad r < R,$$

$$w_{z=0} = \frac{(1+\nu)(3-4\nu)}{\pi E(1-\nu)} p\frac{R}{r} K\left(\frac{R}{r}\right), \qquad r > R,$$

$$v_{z=0} = 0.$$

The expressions for the horizontal displacements are included only for the sake of completeness, for as mentioned we are interested only in the magnitude of the vertical deflections w. Furthermore, in the following we will be occupied only with the deflections of the surface of the elastic medium, for which reason we will let $w = w(r)$ signify the deflections of the surface $z = 0$.

It will be seen that both for case a and case b, we have

$$w(r) = cpK\left(\frac{r}{R}\right), \qquad r < R,$$

$$w(r) = cp\frac{R}{r} K\left(\frac{R}{r}\right), \qquad r > R,$$

provided merely that in case a we put

$$c = \frac{4(1-\nu^2)}{\pi E},$$

while in case b we put

$$c = \frac{(1+\nu)(3-4\nu)}{\pi E(1-\nu)}.$$

FORMULAE FOR DEFLECTIONS AND LOAD

After these introductory considerations, we will now proceed to derive the expression previously referred to for the deflections within a circle of radius R_o, expressed in terms of the deflections exterior to this circle. We will first let the load consist of a uniformly distributed line load p around a circle of radius R, where $R \geqslant R_o$, and where both circles have their centres at the point of origin. The derivation will be carried out in a manner completely analogous to that used by BOUSSINESQ in [2] for the inverse case, where the load operates within a circle, and where the deflections exterior to the circle and the load are expressed in terms of the deflections within the circle.

We now establish

$$f(r) = \int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2 - r^2}},$$

and find for $r > R$

$$\int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = cpR \int_r^\infty \frac{dl}{l\sqrt{l^2-r^2}} \int_0^{\frac{\pi}{2}} \frac{d\omega}{\sqrt{l^2 - R^2\sin^2\omega}}$$

$$= cpR \int_0^{\frac{\pi}{2}} d\omega \int_r^\infty \frac{dl}{l\sqrt{l^2-r^2}\sqrt{l^2-R^2\sin^2\omega}}$$

$$= cpR \int_0^{\frac{\pi}{2}} \frac{d\omega}{2rR\sin\omega} \log\frac{r + R\sin\omega}{r - R\sin\omega}.$$

We now put

$$\alpha = \operatorname{Arcsin}\frac{R}{r},$$

and find

$$\int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = cpR\int_0^{\frac{\pi}{2}}\frac{d\omega}{2rR}\int_0^\alpha \frac{2\cos\varphi}{1-\sin^2\varphi\sin^2\omega}\,d\varphi$$

$$= \frac{cp}{2r}\int_0^\alpha d\varphi\left[2\,\mathrm{Arctg}\,(\cos\varphi\,\mathrm{tg}\,\omega)\right]_{\omega=0}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\frac{cp}{r}\,\mathrm{Arcsin}\,\frac{R}{r}.$$

For $r < R$ we split the integral

$$\int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = \int_r^R + \int_R^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}},$$

and obtain

$$\int_R^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = cpR\int_R^\infty \frac{dl}{l\sqrt{l^2-r^2}}\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{l^2-R^2\sin^2\omega}}$$

$$= cpR\int_0^{\frac{\pi}{2}}d\omega\int_R^\infty \frac{dl}{l\sqrt{l^2-r^2}\sqrt{l^2-R^2\sin^2\omega}}$$

$$= \frac{cp}{r}\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sin\omega}\log\frac{r\sqrt{1-\sin^2\omega}-R\sin\omega\sqrt{1-\left(\frac{r}{R}\right)^2}}{r-R\sin\omega}.$$

In order to simplify here we put

$$\frac{r}{R} = \sin\alpha\,; \quad \sqrt{1-\left(\frac{r}{R}\right)^2} = \cos\alpha,$$

whereupon

$$\int_R^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = \frac{cp}{r}\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sin\omega}\log\frac{\sin(\alpha-\omega)}{\sin\alpha-\sin\omega}$$

$$= \frac{cp}{r}\int_0^{\frac{\pi}{2}}d\omega\int_0^\alpha \frac{1}{\cos\varphi+\cos\omega}\,d\varphi$$

$$= \frac{cp}{r}\int_0^\alpha d\varphi\left[\frac{1}{\sin\varphi}\log\frac{\sin(\varphi-\omega)}{\sin\varphi-\sin\omega}\right]_{\omega=0}^{\frac{\pi}{2}}$$

$$= \frac{cp}{r}\int_0^\alpha \frac{1}{2\sin\varphi}\log\frac{1+\sin\varphi}{1-\sin\varphi}\,d\varphi$$

$$= \frac{cp}{r}\int_0^\alpha d\varphi\left[\frac{1}{2\sin\varphi}\log\frac{1+\sin\varphi\sin\omega}{1-\sin\varphi\sin\omega}\right]_{\omega=0}^{\frac{\pi}{2}}$$

$$= \frac{cp}{r}\int_0^{\frac{\pi}{2}}d\omega\int_0^\alpha \frac{\cos\omega}{1-\sin^2\varphi\sin^2\omega}\,d\varphi$$

$$= \frac{cp}{r}\int_0^{\frac{\pi}{2}}\mathrm{Arctg}\,(\mathrm{tg}\,\alpha\cos\omega)\,d\omega.$$

Further, we find

$$\int_r^R \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = cpR\int_r^R \frac{dl}{l\sqrt{l^2-r^2}}\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{R^2-l^2\sin^2\omega}}$$

$$= cpr\int_0^{\frac{\pi}{2}}d\omega\int_r^R \frac{dl}{l\sqrt{l^2-r^2}\sqrt{R^2-l^2\sin^2\omega}}$$

$$= \frac{cp}{r}\int_0^{\frac{\pi}{2}}\mathrm{Arctg}\,\frac{\sqrt{R^2-r^2}}{r\cos\omega}\,d\omega$$

$$= \frac{cp}{r}\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-\mathrm{Arctg}\,(\mathrm{tg}\,\alpha\cos\omega)\right)d\omega$$

$$= \frac{cp}{r}\frac{\pi^2}{4}-\frac{cp}{r}\int_0^{\frac{\pi}{2}}\mathrm{Arctg}\,(\mathrm{tg}\,\alpha\cos\omega)\,d\omega,$$

whereupon

$$\int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = \frac{cp}{r}\frac{\pi^2}{4}.$$

We now take

$$\frac{d}{dr}r\int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}},$$

and for $r > R$ we obtain

$$\frac{d}{dr} r \int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = \frac{d}{dr}\frac{\pi}{2} cp \operatorname{Arcsin}\frac{R}{r}$$

$$= -\frac{\pi}{2}\frac{cp}{r}\frac{R}{\sqrt{r^2-R^2}},$$

while for $r < R$ we have

$$\frac{d}{dr} r \int_r^\infty \frac{w(l)\,dl}{l\sqrt{l^2-r^2}} = \frac{d}{dr} cp \frac{\pi^2}{4} = 0 .$$

If now $r < R_o < R$, we have, introducing the substitution

$$x = \frac{R}{\sin\omega},$$

$$w(r) = cpR \int_0^{\frac{\pi}{2}} \frac{d\omega}{\sqrt{R^2 - r^2\sin^2\omega}}$$

$$= cpR \int_\infty^R \frac{1}{\sqrt{R^2 - r^2\dfrac{R^2}{x^2}}} \frac{-R}{x\sqrt{x^2-R^2}}\,dx$$

$$= cpR \int_R^\infty \frac{x}{R\sqrt{x^2-r^2}} \frac{R}{x\sqrt{x^2-R^2}}\,dx$$

$$= -\frac{2}{\pi}\int_R^\infty \frac{x\,dx}{\sqrt{x^2-r^2}} \frac{d}{dx} x \int_x^\infty \frac{w(l)\,dl}{l\sqrt{l^2-x^2}}$$

$$= -\frac{2}{\pi}\int_{R_o}^\infty \frac{x\,dx}{\sqrt{x^2-r^2}} \frac{d}{dx} x \int_x^\infty \frac{w(l)\,dl}{l\sqrt{l^2-x^2}}.$$

This formula holds for the deflections produced by any uniformly distributed line load around circles of radius greater than R_o and centre at the point of origin, and must thus also hold for the deflections from any load which is symmetrical about the point of origin and acts exterior to the circle with radius R_o and centre at the point of origin.

By means of a similar formula, the load can be expressed in terms of the deflections $w(l)$ in the loaded area. In order to derive this formula, we assume once again that the load consists of a uniformly distributed line load p around a circle with centre at the point of origin and with a radius R greater than or equal to R_o.

We found above that

$$\frac{d}{dx} x \int_x^\infty \frac{w(l)\,dl}{l\sqrt{l^2-x^2}} = \begin{cases} 0, & x < R, \\ -\dfrac{\pi}{2}\dfrac{cp}{x}\dfrac{R}{\sqrt{x^2-R^2}}, & x > R, \end{cases}$$

and thus for $r < R$ we obtain

$$-\frac{4}{\pi^2 c}\int_{R_o}^r \frac{r\,dx}{\sqrt{r^2-x^2}} \frac{d}{dx} x \int_x^\infty \frac{w(l)\,dl}{l\sqrt{l^2-x^2}} = 0,$$

while for $r > R$ we have

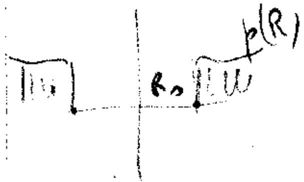
$$-\frac{4}{\pi^2 c}\int_{R_o}^r \frac{r\,dx}{\sqrt{r^2-x^2}} \frac{d}{dx} x \int_x^\infty \frac{w(l)\,dl}{l\sqrt{l^2-x^2}}$$

$$= -\frac{4}{\pi^2 c}\int_R^r \frac{r\,dx}{\sqrt{r^2-x^2}}\left(-\frac{\pi}{2}\frac{cp}{x}\frac{R}{\sqrt{x^2-R^2}}\right)$$

$$= \frac{2}{\pi} prR \int_R^r \frac{dx}{x\sqrt{r^2-x^2}\sqrt{x^2-R^2}}$$

$$= \frac{2}{\pi} prR\left[-\frac{1}{rR}\operatorname{Arctg}\frac{R}{r}\sqrt{\frac{r^2-x^2}{x^2-R^2}}\right]_R^r$$

$$= \frac{2}{\pi} p\left(0 - \left(-\frac{\pi}{2}\right)\right) = p .$$



For an arbitrary, axially symmetric load $p(R)$, exterior to a circle with centre at the point of origin and radius R_o, the condition must consequently hold that

$$\int_{R_o}^r p(R)\,dR = -\frac{4}{\pi^2 c}\int_{R_o}^r \frac{r\,dx}{\sqrt{r^2-x^2}} \frac{d}{dx} x \int_\alpha^\infty \frac{w(l)\,dl}{l\sqrt{l^2-x^2}}.$$

By differentiation with respect to r we obtain from this

$$p(r) = -\frac{4}{\pi^2 c}\frac{d}{dr}\int_{R_o}^r \frac{r\,dx}{\sqrt{r^2-x^2}} \frac{d}{dx} x \int_x^\infty \frac{w(l)\,dl}{l\sqrt{l^2-x^2}}.$$

We have thus demonstrated the following:

If an elastic semi-infinite medium is subjected to a load normal to its surface and symmetric about the point of origin, and acting exterior to a circle with centre at the point of origin and of radius R_0, then the following relations hold for the load $p(r)$ and the deflections $w(r)$:

$$p(r) = -\frac{4}{\pi^2 c}\frac{d}{dr}\int_{R_0}^{r}\frac{r\,dx}{\sqrt{r^2-x^2}}\frac{d}{dx}\,x\int_x^{\infty}\frac{w(l)\,dl}{l\sqrt{l^2-x^2}}, \qquad r > R_0, \qquad (1)$$

and

$$w(r) = -\frac{2}{\pi}\int_{R_0}^{\infty}\frac{x\,dx}{\sqrt{x^2-r^2}}\frac{d}{dx}\,x\int_x^{\infty}\frac{w(l)\,dl}{l\sqrt{l^2-x^2}}, \qquad r < R_0, \qquad (2)$$

both when the surface of the semi-infinite medium is free (case a) and when the surface is constrained with respect to movements parallel to the bounding plane (case b). In (1), the constant c has the values

$$c = \frac{4(1-\nu^2)}{\pi E} \qquad \text{in case } a,$$

and

$$c = \frac{(1+\nu)(3-4\nu)}{\pi E(1-\nu)} \qquad \text{in case } b.$$

ELASTIC MEDIUM WITH BOUNDING WALL

We now turn to our original system, a semi-infinite medium bounded by the rigid wall with circular hole. By using (1) and (2), it is possible to write down integral expressions for the normal components q of the reactions between the semi-infinite medium and the wall, and for the deflections w internal to the hole, for any arbitrary, axially symmetric and vertical load internal to the hole. We establish our coordinate system with point of origin at the centre of the hole and call the radius of the hole R_0. The deflections for the load p of the semi-infinite elastic medium without bounding wall are then

$$w^*(r) = \int_0^r c\,p(R)\,\frac{R}{r}\,K\!\left(\frac{R}{r}\right)dR + \int_r^{R_0} c\,p(R)\,K\!\left(\frac{r}{R}\right)dR, \qquad r < R_0,$$

$$w^*(l) = \int_0^{R_0} c\,p(R)\,\frac{R}{l}\,K\!\left(\frac{R}{l}\right)dR, \qquad l > R_0,$$

and we thus have

$$q(r) = \frac{4}{\pi^2 c}\frac{d}{dr}\int_{R_0}^{r}\frac{r\,dx}{\sqrt{r^2-x^2}}\frac{d}{dx}\,x\int_x^{\infty}\frac{w^*(l)\,dl}{l\sqrt{l^2-x^2}}, \qquad r > R_0, \qquad (3)$$

$$w(r) = w^*(r) + \frac{2}{\pi}\int_{R_0}^{\infty}\frac{x\,dx}{\sqrt{x^2-r^2}}\frac{d}{dx}\,x\int_x^{\infty}\frac{w^*(l)\,dl}{l\sqrt{l^2-x^2}}, \qquad r < R_0. \qquad (4)$$

These two formulae solve our problem, and we will now use them on some cases of loading.

Example 1

We will first examine a vertical, single force P at the point of origin (see fig. 2). Here we have

$$w^*(l) = \frac{c}{4}P\frac{1}{l}, \qquad l > 0.$$

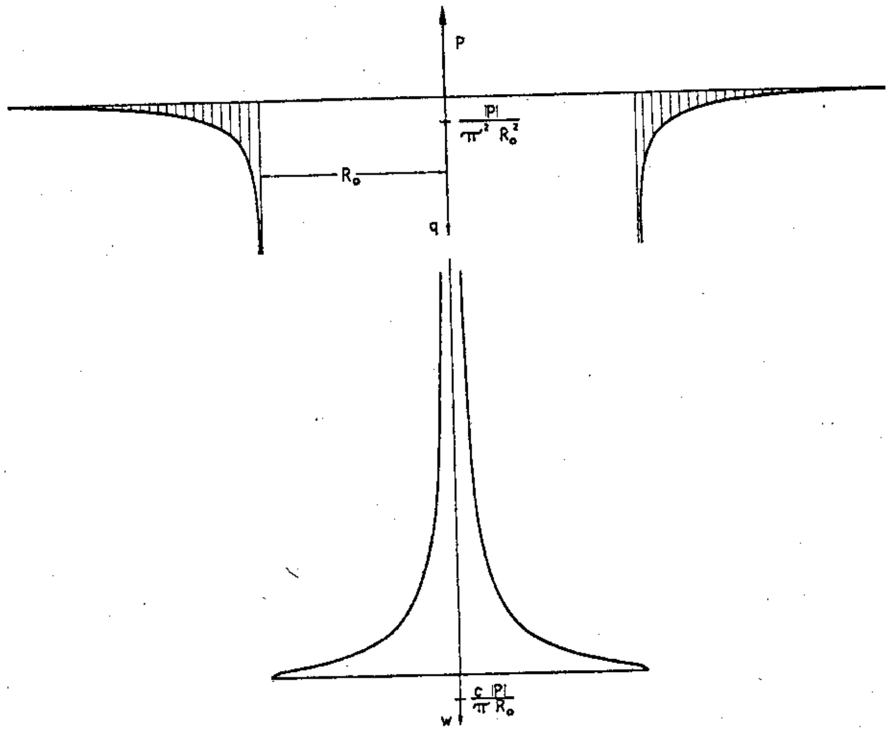


Fig. 2. Vertical reactions q and deflections w for a single force.

We insert this expression in (3) and (4), and find

$$\int_x^\infty \frac{\frac{1}{l}\,dl}{l\sqrt{l^2-x^2}} = \left[\frac{\sqrt{l^2-x^2}}{x^2 l}\right]_x^\infty = \frac{1}{x^2},$$

and accordingly

$$\frac{d}{dx}\, x \int_x^\infty \frac{\frac{1}{l}\,dl}{l\sqrt{l^2-x^2}} = -\frac{1}{x^2}.$$

The reactions q then become

$$\begin{aligned} q(r) &= \frac{4}{\pi^2 c}\,\frac{c}{4}\,P\,\frac{d}{dr}\int_{R_o}^{r}\frac{r\,dx}{\sqrt{r^2-x^2}}\left(-\frac{1}{x^2}\right) \\ &= -\frac{P}{\pi^2}\,\frac{d}{dr}\, r\left[-\frac{\sqrt{r^2-x^2}}{r^2 x}\right]_{R_o}^{r} \\ &= -\frac{P}{\pi^2}\,\frac{d}{dr}\,\frac{\sqrt{r^2-R_o^2}}{r R_o} \\ &= -\frac{P}{\pi^2}\,\frac{R_o}{r^2\sqrt{r^2-R_o^2}}, \qquad r > R_o. \end{aligned}$$

The deflections w become

$$\begin{aligned} w(r) &= \frac{c}{4}\,P\,\frac{1}{r} + \frac{2}{\pi}\,\frac{c}{4}\,P\int_{R_o}^{\infty}\frac{x\,dx}{\sqrt{x^2-r^2}}\left(-\frac{1}{x^2}\right) \\ &= \frac{c}{4}\,P\,\frac{1}{r} - \frac{2}{\pi}\,\frac{c}{4}\,P\left[\frac{1}{r}\,\mathrm{Arccos}\,\frac{r}{x}\right]_{R_o}^{\infty} \\ &= \frac{c}{4}\,P\,\frac{1}{r}\left(1-\frac{2}{\pi}\left(\frac{\pi}{2}-\mathrm{Arccos}\,\frac{r}{R_o}\right)\right) \\ &= \frac{c}{2\pi}\,P\,\frac{1}{r}\,\mathrm{Arccos}\,\frac{r}{R_o}, \qquad r < R_o. \end{aligned}$$

Example 2

The next example of loading will be a uniformly distributed line load p around a circle with centre at the point of origin and radius $R < R_o$ (see fig. 3).

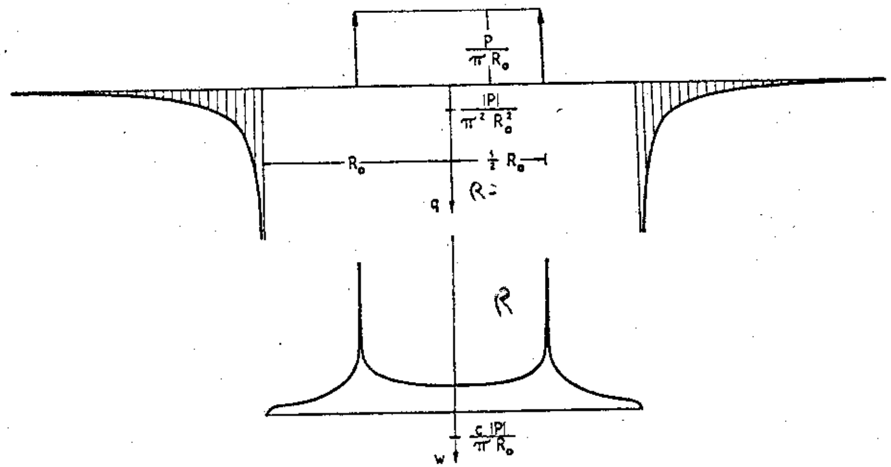


Fig. 3. Vertical reactions q and deflections w for a circular line load.

In this case we have

$$w^*(l) = cp\,\frac{R}{l}\,K\!\left(\frac{R}{l}\right) = cpR\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{l^2-R^2\sin^2\pi}}, \qquad l > R_o,$$

and we found earlier (page 100)

$$\begin{aligned} \frac{d}{dx}\, x\int_x^\infty \frac{w^*(l)\,dl}{l\sqrt{l^2-x^2}} &= cpR\,\frac{d}{dx}\, x\int_x^\infty \frac{dl}{l\sqrt{l^2-x^2}}\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{l^2-R^2\sin^2\omega}} \\ &= -\frac{\pi}{2}\,\frac{cp}{x}\,\frac{R}{\sqrt{x^2-R^2}}. \end{aligned}$$

The reactions q therefore become

$$\begin{aligned} q(r) &= \frac{4}{\pi^2 c}\,\frac{d}{dr}\int_{R_o}^{r}\frac{r\,dx}{\sqrt{r^2-x^2}}\left(-\frac{\pi}{2}\,\frac{cp}{x}\,\frac{R}{\sqrt{x^2-R^2}}\right) \\ &= -\frac{2}{\pi}\,pR\,\frac{d}{dr}\, r\int_{R_o}^{r}\frac{dx}{x\sqrt{r^2-x^2}\sqrt{x^2-R^2}} \\ &= -\frac{2}{\pi}\,pR\,\frac{d}{dr}\, r\left[-\frac{1}{Rr}\,\mathrm{Arctg}\,\frac{R}{r}\sqrt{\frac{r^2-x^2}{x^2-R^2}}\right]_{R_o}^{r} \end{aligned}$$

$$= -\frac{2}{\pi}\, p\, \frac{d}{dr}\, \text{Arctg}\, \frac{R}{r}\sqrt{\frac{r^2-R_o^2}{R_o^2-R^2}}$$

$$= -\frac{2}{\pi}\, p\, \frac{R}{r^2-R^2}\sqrt{\frac{R_o^2-R^2}{r^2-R_o^2}}\,, \qquad r > R_o\,.$$

The deflections from the reactions q become

$$\frac{2}{\pi}\int_{R_o}^{\infty}\frac{x\,dx}{\sqrt{x^2-r^2}}\,\frac{d}{dx}\,x\int_{x}^{\infty}\frac{w^*(l)\,dl}{l\sqrt{l^2-x^2}}$$

$$= \frac{2}{\pi}\int_{R_o}^{\infty}\frac{x\,dx}{\sqrt{x^2-r^2}}\left(-\frac{\pi}{2}\,\frac{cp}{x}\,\frac{R}{\sqrt{x^2-R^2}}\right)$$

$$= -cpR\int_{R_o}^{\infty}\frac{dx}{\sqrt{x^2-r^2}\sqrt{x^2-R^2}}$$

$$= \begin{cases} -cp\displaystyle\int_{\frac{R}{x}=0}^{\frac{R}{R_o}}\frac{d\frac{R}{x}}{\sqrt{1-\left(\frac{R}{x}\right)^2}\sqrt{1-\left(\frac{r}{R}\right)^2\left(\frac{R}{x}\right)^2}}, & r<R \\[2ex] -cp\,\dfrac{R}{r}\displaystyle\int_{\frac{r}{x}=0}^{\frac{r}{R_o}}\frac{d\frac{r}{x}}{\sqrt{1-\left(\frac{r}{x}\right)^2}\sqrt{1-\left(\frac{R}{r}\right)^2\left(\frac{r}{x}\right)^2}}, & r>R \end{cases}$$

$$= \begin{cases} -cpF\left(\dfrac{R_o}{R},\dfrac{r}{R}\right), & r<R \\[2ex] -cp\,\dfrac{R}{r}\,F\left(\dfrac{r}{R_o},\dfrac{R}{r}\right), & r>R\,. \end{cases}$$

The integrals designated by $F(a,k)$ are elliptic integrals of the 1st kind.

The resulting deflections w become

$$w(r) = \begin{cases} cp\left(K\left(\dfrac{r}{R}\right)-F\left(\dfrac{R}{R_o},\dfrac{r}{R}\right)\right), & r<R\,, \\[2ex] cp\,\dfrac{R}{r}\left(K\left(\dfrac{R}{r}\right)-F\left(\dfrac{r}{R_o},\dfrac{R}{r}\right)\right), & r>R\,. \end{cases}$$

It will be observed from the formula for the reactions q that they have the same sign throughout, from which it is deduced that an arbitrary, axially symmetric loading, which is a normal tension throughout, must give rise to compressive stresses everywhere between wall and semi-infinite medium.

Example 3

On the basis of the formulae for reactions and deflections arising from a circular line load, it is possible to find these values for any arbitrary axially symmetric loading $p(R)$ by integrating with respect to R.

We will perform here the integrations for the case where $p(R)$ is constant for $R \leq \alpha R_o$ $(\alpha \leq 1)$ and zero for $\alpha R_o < R \leq R_o$ that is, for a uniformly distributed load within a circle whose centre is at the point of origin and whose radius is αR_o (see fig. 4).

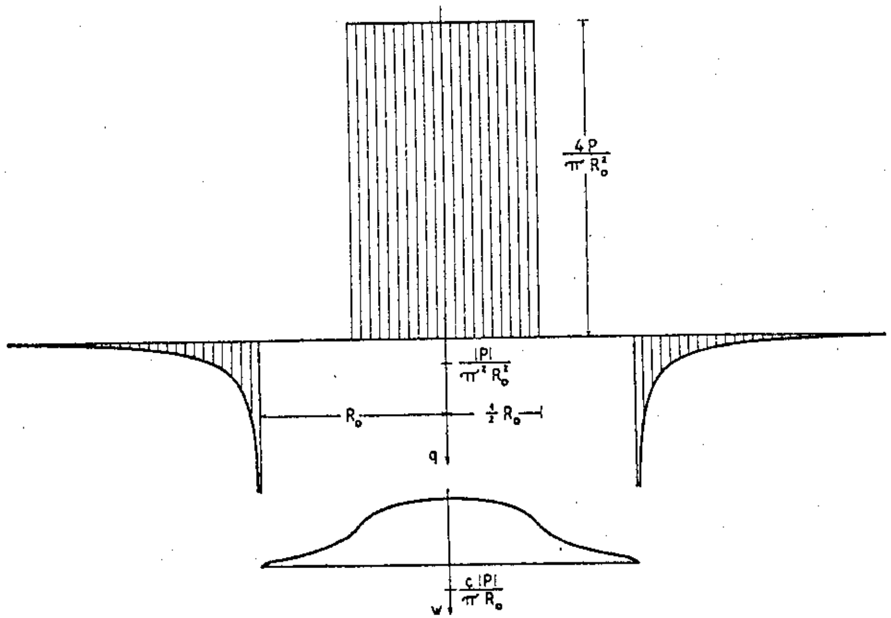


Fig. 4. Vertical reactions q and deflections w for a uniformly distributed load within a circle.

The reactions q become

$$q(r) = \int_0^{\alpha R_o} -\frac{2}{\pi}\, p\, \frac{R}{r^2-R^2}\sqrt{\frac{R_o^2-R^2}{r^2-R_o^2}}\, dR$$

$$= -\frac{p}{\pi}\,\frac{1}{\sqrt{r^2-R_o^2}}\int_{R=0}^{\alpha R_o}\frac{\sqrt{R_o^2-R^2}}{r^2-R^2}\, dR^2$$

$$= -\frac{p}{\pi}\,\frac{1}{\sqrt{r^2-R_o^2}}\left[-2\sqrt{R_o^2-R^2}+2\sqrt{r^2-R_o^2}\,\mathrm{Arctg}\sqrt{\frac{R_o^2-R^2}{r^2-R_o^2}}\right]_{R=0}^{\alpha R_o}$$

$$= -\frac{2p}{\pi}\,\frac{1}{\sqrt{r^2-R_o^2}}\left\{R_o - R_o\sqrt{1-\alpha^2}-\sqrt{r^2-R_o^2}\left(\mathrm{Arctg}\sqrt{\frac{R_o^2}{r^2-R_o^2}} - \mathrm{Arctg}\sqrt{\frac{R_o^2(1-\alpha^2)}{r^2-R_o^2}}\right)\right\}$$

$$= -\frac{2p}{\pi}\left\{\frac{1-\sqrt{1-\alpha^2}}{\sqrt{\left(\frac{r}{R_o}\right)^2-1}} - \mathrm{Arctg}\,\frac{1}{\sqrt{\left(\frac{r}{R_o}\right)^2-1}} + \mathrm{Arctg}\sqrt{\frac{1-\alpha^2}{\left(\frac{r}{R_o}\right)^2-1}}\right\}$$

$$= -\frac{2p}{\pi}\left\{\frac{1-\sqrt{1-\alpha^2}}{\sqrt{\left(\frac{r}{R_o}\right)^2-1}} - \mathrm{Arctg}\,\frac{\sqrt{\left(\frac{r}{R_o}\right)^2-1}\,\left(1-\sqrt{1-\alpha^2}\right)}{\left(\frac{r}{R_o}\right)^2-1+\sqrt{1-\alpha^2}}\right\}.$$

In the following expression for the deflections, elliptic integrals of the 2nd kind are involved:

$$E(b,k) = \int_o^b \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\, dx\,, \qquad b \leq 1\,,$$

and

$$E(k) = E(1,k)\,.$$

In the subsequent integration, we make use of the fact that

$$\frac{\delta}{\delta k}\left(\frac{1}{k}E\left(\gamma\frac{1}{k},k\right)\right) = -\frac{1}{k^2}\left\{F\left(\gamma\frac{1}{k},k\right)+\frac{\sqrt{1-\gamma^2}}{\sqrt{\left(\frac{k}{\gamma}\right)^2-1}}\right\},$$

and accordingly

$$\int \frac{1}{k^2}F\left(\gamma\frac{1}{k},k\right)dk = -\frac{1}{k}E\left(\gamma\frac{1}{k},k\right) - \frac{\sqrt{1-\gamma^2}}{\gamma}\sqrt{1-\left(\frac{\gamma}{k}\right)^2}\,,$$

and of the fact that

$$\frac{\delta}{\delta k}\left(E(\beta,k)-(1-k^2)F(\beta,k)\right) = k\,F(\beta,k)+\frac{k\beta\sqrt{1-\beta^2}}{\sqrt{1-k^2\beta^2}}\,,$$

and accordingly

$$\int kF(\beta,k)\,dk = E(\beta,k)-(1-k^2)F(\beta,k)+\sqrt{1-k^2\beta^2}\,\frac{\sqrt{1-\beta^2}}{\beta}\,.$$

The deflections w thus become, for $R_o > r > \alpha R_o$

$$w(r) = \int_{R=0}^{\alpha R_o} cp\,\frac{R}{r}\left(K\left(\frac{R}{r}\right)-F\left(\frac{r}{R_o},\frac{R}{r}\right)\right)dR$$

$$= cpr\int_{R=0}^{\alpha R_o}\frac{R}{r}\left\{K\left(\frac{R}{r}\right)-F\left(\frac{r}{R_o},\frac{R}{r}\right)\right\}d\frac{R}{r}$$

$$= cpr\left[E\left(\frac{R}{r}\right)-\left(1-\left(\frac{R}{r}\right)^2\right)K\left(\frac{R}{r}\right)-\left\{E\left(\frac{r}{R_o},\frac{R}{r}\right)-\left(1-\left(\frac{R}{r}\right)^2\right)F\left(\frac{r}{R_o},\frac{R}{r}\right)+\sqrt{1-\left(\frac{R}{R_o}\right)^2}\,\frac{\sqrt{1-\left(\frac{r}{R_o}\right)^2}}{\frac{r}{R_o}}\right\}\right]_{R=0}^{\alpha R_o}$$

$$= cpr\left\{E\left(\frac{\alpha R_o}{r}\right)-E\left(\frac{r}{R_o},\frac{\alpha R_o}{r}\right)-\left[1-\left(\frac{\alpha R_o}{r}\right)^2\right]\left[K\left(\frac{\alpha R_o}{r}\right)-F\left(\frac{r}{R_o},\frac{\alpha R_o}{r}\right)\right]+\left(1-\sqrt{1-\alpha^2}\right)\sqrt{\left(\frac{R_o}{r}\right)^2-1}\right\},$$

and for $\alpha R_o > r > 0$

$$w(r) = \int_0^r cp\frac{R}{r}\left[K\left(\frac{R}{r}\right) - F\left(\frac{r}{R_0}, \frac{R}{r}\right)\right] dR + \int_r^{\alpha R_0} cp\left[K\left(\frac{r}{R}\right) - F\left(\frac{R}{R_0}, \frac{r}{R}\right)\right] dR$$

$$= \int_{R=0}^r cpr\frac{R}{r}\left[K\left(\frac{R}{r}\right) - F\left(\frac{r}{R_0}, \frac{R}{r}\right)\right] d\frac{R}{r} - \int_{R=r}^{\alpha R_0} cpr\frac{R^2}{r^2}\left[K\left(\frac{r}{R}\right) - F\left(\frac{R}{R_0}, \frac{r}{R}\right)\right] d\frac{r}{R}$$

$$= cpr\Biggl\{\Biggl[E\left(\frac{R}{r}\right) - \left(1-\left(\frac{R}{r}\right)^2\right)K\left(\frac{R}{r}\right) - \Biggl\{E\left(\frac{r}{R_0}, \frac{R}{r}\right) - \left(1-\left(\frac{R}{r}\right)^2\right)F\left(\frac{r}{R_0}, \frac{R}{r}\right) \cdot$$

$$+ \sqrt{1-\left(\frac{R}{R_0}\right)^2}\sqrt{\left(\frac{R_0}{r}\right)^2 - 1}\Biggr\}\Biggr]_{R=0}^{r}$$

$$+ \left[\frac{R}{r}E\left(\frac{r}{R}\right) - \frac{R}{r}E\left(\frac{R}{R_0}, \frac{r}{R}\right) - \sqrt{\left(\frac{R_0}{r}\right)^2 - 1}\sqrt{1-\left(\frac{r}{R_0}\right)^2\left(\frac{R}{r}\right)^2}\right]_{R=r}^{\alpha R_0}\Biggr\}$$

$$= cpr\Biggl\{1 - \frac{r}{R_0} + \left(1 - \sqrt{1-\left(\frac{r}{R_0}\right)^2}\right)\sqrt{\left(\frac{R_0}{r}\right)^2 - 1}$$

$$+ \frac{\alpha R_0}{r}E\left(\frac{r}{\alpha R_0}\right) - \frac{\alpha R_0}{r}E\left(\alpha, \frac{r}{\alpha R_0}\right) - \sqrt{\left(\frac{R_0}{r}\right)^2 - 1}\sqrt{1-\alpha^2}$$

$$- 1 + \frac{r}{R_0} + \sqrt{\left(\frac{R_0}{r}\right)^2 - 1}\sqrt{1-\left(\frac{r}{R_0}\right)^2}\Biggr\}$$

$$= cpr\left\{\frac{\alpha R_0}{r}E\left(\frac{r}{\alpha R_0}\right) - \frac{\alpha R_0}{r}E\left(\alpha, \frac{r}{\alpha R_0}\right) + \sqrt{\left(\frac{R_0}{r}\right)^2 - 1}\left(1-\sqrt{1-\alpha^2}\right)\right\}$$

$$= cpR_0\left\{\sqrt{1-\left(\frac{r}{R_0}\right)^2}\left(1-\sqrt{1-\alpha^2}\right) + \alpha\left(E\left(\frac{r}{\alpha R_0}\right) - E\left(\alpha, \frac{r}{\alpha R_0}\right)\right)\right\}.$$

For $\alpha = 1$, that is to say, for a loading p uniformly distributed over the hole, we obtain the following simple expressions

$$q(r) = -\frac{2p}{\pi}\left(\frac{1}{\sqrt{\left(\frac{r}{R_0}\right)^2 - 1}} - \text{Arctg}\frac{1}{\sqrt{\left(\frac{r}{R_0}\right)^2 - 1}}\right), \qquad r > R_0 .$$

and

$$w(r) = cpR_0\sqrt{1-\left(\frac{r}{R_0}\right)^2}, \qquad r < R_0 .$$

For an arbitrary α, the deflection at the point of origin becomes

$$w(0) = cpR_0\left(1 - \sqrt{1-\alpha^2} + \alpha \text{ Arccos } \alpha\right),$$

and the deflection at the edge of the loaded area becomes

$$w(\alpha R_0) = cpR_0\left(\sqrt{1-\alpha^2} - 1 + \alpha\right).$$

The results obtained above have been employed in studies by V. ASKEGAARD [1] and S. GRAVESEN [3] in connection with an investigation of the measurement of pressure between an elastic medium and a rigid wall.

REFERENCES

[1] ASKEGAARD, V. *The measurement of pressure between an infinitely rigid wall and a compressible medium.* Festskrift til professor Anker Engelund. Meddelelse nr. 10 fra Lab. for Bygningstekn. 1959.

[2] BOUSSINESQ, J. *Application des potentiels....* Paris 1885.

[3] GRAVESEN, S. *Elastisk halvrum begrænset af en stiv plade med cirkulært stempel.* Festskrift til professor Anker Engelund. Meddelelse nr. 10 fra Lab. for Bygningstekn. 1959.

[4] TIMOSHENKO, S. *Theory of Elasticity.* New York 1934.

[5] WALÉN, C. *Et fall av deformation i halvoändeligt elastisk medium.* Teknisk Tidsskrift 72, 1942, p. 383–85.

SUMMARY

The first part of this paper deals with a semi-infinite elastic medium, subjected to an axially symmetric load exterior to a circle. Integral expressions are derived for the load (1) and for the deflections in the unloaded area (2) both in terms of the deflections in the loaded area. Next an elastic medium bounded by a rigid wall with a circular hole is considered, and integral expressions are given for the reactions betweens wall and elastic medium (3) and for the deflections inside the hole (4) for arbitrary axially symmetric loading inside the hole. Formulae (3) and (4) are used on the following cases of loading: a single force, a circular line load, and a load uniformly distributed inside a circle.

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Contributions to the discussion of the above paper can be sent to the editor till the 28th February 1960. Published contributions will be entered on the table of contents for the following volume.

ADDENDUM

A paper by I. N. Sneddon [2] which gives a solution of the problems discussed in [1] has come to my knowledge after the printing of my paper.

The methods used in [2] are different from those of [1] and the solution also has another form. Using the notation of [1] the normal component of the surface displacement is given by the equation

$$w(r) = \frac{4(1-\nu^2)}{\pi E} R_o \int_{\frac{r}{R_o}}^{1} \frac{\mu\, d\mu}{\sqrt{\mu^2 - \left(\frac{r}{R_o}\right)^2}} \int_0^1 \frac{x\, q(x\mu R_o)}{\sqrt{1-x^2}} dx, \quad r < R_o$$

according to formula (3.2.4.) in [2].

A solution is also given for the case where the applied pressure p is constant over a circular area of radius $\alpha R_o (\alpha < 1)$, namely:

$$w(r) = \frac{4(1-\nu^2)}{\pi E} p \sqrt{R_o^2 - r^2} \left\{1 - \sqrt{1-\alpha^2}\right\}, \text{ (3.3.2.) in [2].}$$

This expression is not in agreement with the result obtained in [1] and must be wrong. In fact, inserting

$$q(r) = \begin{cases} p, & 0 < r < \alpha R_o, \\ 0, & \alpha R_o < r < R_o \end{cases}$$

in (3.2.4.) we get for $\varrho = \frac{r}{R_o} > \alpha$

$$\int_0^1 \frac{x\, q(x\mu R_o)}{\sqrt{1-x^2}} dx = \int_0^{\frac{\alpha}{\mu}} \frac{x\, p\, dx}{\sqrt{1-x^2}} = p\left\{1 - \sqrt{1 - \frac{\alpha^2}{\mu^2}}\right\}, \quad \mu > \alpha$$

and

$$w = \frac{4(1-\nu^2)}{\pi E} p R_o \int_\varrho^1 \frac{\mu\, d\mu}{\sqrt{\mu^2 - \varrho^2}} \left(1 - \sqrt{1 - \frac{\alpha^2}{\mu^2}}\right).$$

If now $\frac{\varrho}{\mu} = t$ and $\frac{\alpha}{\varrho} = k$ we have

$$w = \frac{4(1-\nu^2)}{\pi E} p R_o \int_\varrho^1 \frac{1 - \sqrt{1-k^2t^2}}{t^2\sqrt{1-t^2}} dt,$$

$$w = \frac{4(1-\nu^2)}{\pi E} pr \int_\varrho^1 \left\{ \sqrt{\frac{1-k^2t^2}{1-t^2}} - (1-k^2)\frac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} + \frac{\sqrt{1-k^2t^2} - 1 + k^2t^4}{t^2\sqrt{1-t^2}\sqrt{1-k^2t^2}} \right\} dt$$

$$= \frac{4(1-\nu^2)}{\pi E} pr \left\{ E(k) - E(\varrho,k) - (1-k^2)(K(k) - F(\varrho,k)) + \frac{1}{\varrho}\left(1 - \sqrt{1-\alpha^2}\right)\sqrt{1-\varrho^2} \right\},$$

which is identical with the result obtained in [1]. Also for $0 < \varrho = \frac{r}{R_o} < \alpha$ we get the same equation as in [1]:

$$w = \frac{4(1-\nu^2)}{\pi E} p R_o \left\{ \alpha \left[E\left(\frac{\varrho}{\alpha}\right) - E\left(\alpha, \frac{\varrho}{\alpha}\right) \right] + \left(1 - \sqrt{1-\alpha^2}\right)\sqrt{1-\varrho^2} \right\}.$$

REFERENCES

[1] Gravesen, S. *Elastic semi-infinite medium bounded by a rigid wall with a circular hole.* Bygningsstatiske Meddelelser, vol. 30, 1959, pp. 93–111.

[2] Sneddon, I. N. *The distribution of stress in the neighbourhood of a crack in an elastic solid.* Proc. Royal Soc. London, Series A, vol. 187, 1946, pp. 229–260.